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LEBESGUE-RADON-NIKODYM TYPE THEOREMS FOR OPERATORS DEFINED ON ORDERED BANACH SPACES

BY

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The aim of this note is to give Lebesgue-Radon-Nikodym type theorems for the notion of absolute continuity defined below.

In [4] we introduced the following relation between bounded linear operators $U : Z \to X$ (here Z denotes an ordered Banach space and X a Banach space) and positive functionals $\mu \in Z^*$:

DEFINITION. U is said to be locally absolutely continuous with respect to μ (i.e. $U \ll \mu$) if for every $\varepsilon > 0$ and every $z \in Z$, $z > 0$, there exists a $\delta = \delta(\varepsilon, z) > 0$ such that :

$$0 < y < z, \ \mu(y) < \delta, \text{ implies } \| U(y) \| < \varepsilon.$$

It was remarked by Bourbaki [1] that for μ and λ two positive Radon measures given on a compact Hausdorff space S the following statements are equivalent :

(i) $\mu \ll \lambda$ in the sense of the definition above

(ii) $\mu \ll \lambda$ as measures defined on the Borel σ-algebra. $\mathscr{B}(S)$ associated to S, i.e., for every $\varepsilon > 0$ and every $A \in \mathscr{B}(S)$ there exists a $\delta = \delta(\varepsilon, A) > 0$ such that :

$$B \in \mathscr{B}(S), \ B \subset A, \ \lambda(B) < \delta \text{ implies } \mu(B) < \varepsilon$$

A more precise result was obtained in [5] (see also [6] for details) where the following operational analogue of the well known theorem of Bartle-Dunford-Schwartz concerning the existence of control measures is proved : "Let S be a compact Hausdorff space and let X be a Banach space. Then an operator $T \in \mathscr{L}(C(S), X)$ is weakly compact if and only if there exists a positive Radon measure μ on S such that $T \ll \mu$."

Our results make use of methods from the theory of ordered vector spaces, a key role being played by the Freudenthal theorem on spectral representation in the form given by Yoshida [9]. Because the natural order on a w^*-algebra fails to have Riesz decomposition property, the theory developed here cannot be applied in such a situation. On the other hand our non-commutative extension for Lebesgue-Radon-Nikodym theorem (see Theorem 2.4) is formally identical to the result obtained by

Sakai [8] in the case of w^*-algebras. So, we strongly believe that both these results can be derived from a general Lebesgue-Radon-Nicodym type theorem.

1. REVIEW ON ORDERED VECTOR SPACES

The aim of this section is to recall some basic definitions and results which will be needed.

An ordered Banach space X is a Riesz space if the positive cone is closed and has the Riesz decomposition property. If X^* denotes the topological dual of X then, as well known, X^* is an order complete Banach lattice.

A positive element of an order σ-complete Banach lattice Y is said to be a Freudenthal unit if $\inf(u, |y|) = 0$ implies that $y = 0$. In this case each $e \in Y$ with $\inf(e, u - e) = 0$ is called a quasi-unit and the elements belonging to the linear hull of all quasi-units are usually called simple. The following result is due to Yoshida [9]:

1.1 THEOREM. *Let Y be an order σ-complete Banach lattice with unit. Then every positive element of Y is the least upper bound (l.u.b.) of an increasing sequence of positive simple elements.*

Another important property of Y is that each $y \in Y$ generates a band projection $P_y : Y \to Y$ whose image is

$$[y] = \{z \in Y ; \inf(|z|, |x|) = 0 \text{ for all } x \in Y \text{ with } \inf(|x|, |y|) = 0\}.$$

A positive functional $y^* \in Y^*$ is said to be order σ-continuous if $y_n \downarrow 0$ (in order) implies $y^*(y_n) \to 0$. We shall denote by Y_o^* the vector sublattice of all $y^* \in Y^*$ which are the difference of two order σ-continuous positive functionals.

1.2 PROPOSITION. Let $\mu \in Y_o^*$, $\mu > 0$, and let $\lambda \in Y^*$. The following statements are equivalent:

(a) $\lambda \in [\mu]$,

(b) $\lambda \ll \mu$ i.e. for each $z \in Y$, $z > 0$ and each $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon, z) > 0$ such that:

$$|y| \leqslant z, \quad \mu(|y|) < \delta \text{ implies } \quad |\lambda(y)| < \varepsilon$$

(c) $\lambda \in Y_o^*$ and in addition:

$$y > 0, \quad \mu(y) = 0 \text{ implies } \quad \lambda(y) = 0.$$

Proof. The equivalence (a) $\Leftrightarrow$ (b) was remarked by Bourbaki [1] ch. 2, Proposition 4. We shall show only that (c) $\Rightarrow$ (b). Suppose that the contrary is true. Then there exist a $z_0 \in Y$, $z_0 > 0$, a scalar $\varepsilon_0 > 0$ and a sequence $z_n \in [0, z_0]$ such that $\mu(z_n) < 2^{-n}$ and $|\lambda|(z_n) > \varepsilon_0$ for all $n \geqslant 1$. Put :

$$z = \inf \{\sup (z_k ; k \geqslant n) ; n \geqslant 1\}.$$

Since λ and μ are order σ-continuous it follows that $\mu(z) = 0$ and $|\lambda|(z) > \varepsilon_0$. By hypothesis, we have that $\lambda(y) = 0$ for each $0 \leqslant y \leqslant z$ and thus :

$$\lambda_+(z) = \sup \{\lambda(y) ; 0 \leqslant y \leqslant z\} = 0.$$

Hence $|\lambda|(z) = 2\lambda_+(z) - \lambda(z) = 0$, which contradicts the fact that $\varepsilon_0 \neq 0$, q.e.d.

We shall need also the following extension property for positive functionals :

1.3 PROPOSITION. *Let X be a linear subspace of a Riesz space Y. If $g \in Y^*$, $g > 0$, then each linear functional $f : X \to R$ satisfying the inequality :*

$$f(x) \leqslant \inf \{g(y) ; y \in Y, \ y \geqslant x, 0\}$$

for all $x \in X$, has an extension $h \in Y^*$ with $0 \leqslant h \leqslant g$.

This result is an easy consequence of the Hahn-Banach theorem.

We end this section by discussing a new concept that seems to be the natural framework for studying the abstract Lebesgue-Radon-Nikodym type theorems.

1.4 DEFINITION An ordered quasi-algebra with unit $u > 0$ is a triplet $(\mathscr{A}, *, \tilde{\mathscr{A}})$ that satisfies the following four conditions :

(Q1) $\mathscr{A}$ and $\tilde{\mathscr{A}}$ are Riesz spaces and $\tilde{\mathscr{A}}$ contains $\mathscr{A}$ as an ordered vector subspace ;

(Q2) $* : \mathscr{A} \times \mathscr{A} \to \tilde{\mathscr{A}}$ is a positive bilinear mapping

(Q3) $a * u = u * a = a$ for every a ;

(Q4) $\mathscr{A}$ is contained in the band generated by u in $\mathscr{A}^{**}$ (i.e., u is total over $\mathscr{A}$).

A non trivial exemple can be obtained by considering a positive Radon measure, say μ, given on a compact space. Then $(L^2(\mu), *, L^1(\mu))$ is an ordered quasi-algebra with unit if one considers for $*$, the pointwise multiplication.

1.5 DEFINITION. Given two quasi-algebras with unit, say $Q = (\mathscr{A}, *, \tilde{\mathscr{A}})$ and $Q' = (\mathscr{A}', *', \tilde{\mathscr{A}}')$, a morphism from Q into Q' is a pair $(\varphi, \tilde{\varphi})$, where :

(i) $\varphi \in \mathscr{L}(\mathscr{A}, \mathscr{A}')$,

(ii) $\tilde{\varphi} \in \mathscr{L}(\tilde{\mathscr{A}}, \tilde{\mathscr{A}}')$,

(iii) $\tilde{\varphi}(a * b) = \varphi(a) *' \varphi(b)$ for all $a, b \in \mathscr{A}$,

(iv) $\tilde{\varphi}(u) = u'$.

According to the above definition Q is a sub-quasi-algebra of Q' if $\mathscr{A}$ is an ordered subspace of $\mathscr{A}'$, $\tilde{\mathscr{A}}$ is an ordered subspace of $\tilde{\mathscr{A}}'$, $u = u'$ and $a * b = a *' b$ for all $a, b \in \mathscr{A}$.

If $\mathscr{A}$ is a Banach lattice and also an ordered Banach algebra with multiplicative unit $u > 0$, then every closed sublattice $\mathscr{H}$ of $\mathscr{A}$ gives rise to a sub-quasi-algebra $(\mathscr{H}, *, \mathscr{A})$ provided that $u \in \mathscr{H}$ and $\mathscr{H} \subset [u]$, the band being calculated in $\mathscr{A}^{**}$.

The order properties of a quasi-algebra $Q = (\mathscr{A}, *, \tilde{\mathscr{A}})$ can be improved by embedding Q into its second dual. Given an $a \in \mathscr{A}$ we can consider the following two operators $L_a, R_a \in \mathscr{L}(\mathscr{A}, \tilde{\mathscr{A}})$ defined by $L_a(z) = a * z$ and $R_a(z) = z * a$, $z \in \mathscr{A}$. Then the mappings $a \to L_a^*$ and $a \to R_a^*$ can be extended to $\mathscr{A}^{**}$ as follows:

$$(L_x^* f)a = x(R_a^* f)$$

and

$$(R_x^* f)a = x(L_a^* f)$$

for all $x \in \mathscr{A}^{**}$, $f \in \tilde{\mathscr{A}}^*$, $a \in \mathscr{A}$. Put:

$$x *_\rho y = L_x^{**}(y)$$

and

$$x *_\lambda y = R_x^{**}(y)$$

for all $x, y \in \mathscr{A}^{**}$. Then $(\mathscr{A}^{**}, *_\rho, \tilde{\mathscr{A}}^{**})$ and $(\mathscr{A}^{**}, *_\lambda, \tilde{\mathscr{A}}^{**})$ both satisfy (Q1) — (Q3) above. For (Q4) we must consider $[u]$, the band generated by u in A^{**}, instead of the entire space A^{**}. Thus we obtain two order complete quasi-algebras, both containing Q as a sub-quasi-algebra.

2. THE MAIN RESULTS

The aim of this section is to describe the band generated by a positive functional μ in the special case when μ is defined on an ordered quasi-algebra $Q = (\mathscr{A}, *, \tilde{\mathscr{A}})$ with unit $u > 0$.

Let $\tilde{\mu} \in \tilde{\mathscr{A}}^*$, $\tilde{\mu} > 0$ and put $\mu = \tilde{\mu} \mid \mathscr{A}$.

2.1 LEMMA. *If $\mathscr{A}$ is order σ-complete and the bilinear mapping $(a, b) \to \tilde{\mu}(a * b)$ is order σ-continuous in each argument then:*

$$a \in \mathscr{A},\ \mu(|a|) = 0 \text{ implies } \tilde{\mu}(a * b) = \tilde{\mu}(b * a)\ 0 \text{ for every } b \in \mathscr{A}.$$

Consequently (see Proposition 1.2 above), the functionals $\tilde{\mu} \circ L_a$ and $\tilde{\mu} \circ R_a$ both belong to $[\mu]$ for all $a \in \mathscr{A}$.

Proof. Clearly, the implication is true for every $b \in \mathscr{A}$ with $|b| \leqslant \gamma a$, particularly for all simple elements of $\mathscr{A}$. The case of an arbitrary $b \in \mathscr{A}$ follows now from Theorem 1.1 above, q.e.d.

2.2 LEMMA. *If $\mathscr{A}$ is order σ-complete and the bilinear mapping $(a, b) \to \tilde{\mu}(a * b)$ is order σ-continuous in each argument, then for each*

$\nu \in \mathcal{A}^*$ with $\inf(\nu, \mu - \nu) = 0$ there exists an $e \in \mathcal{A}$ which satisfies the following three conditions:

i) $0 \leqslant e \leqslant u$,

ii) $\mu(\inf(e, u - e)) = 0$,

iii) $\nu = \tilde{\mu} \circ L_e = \tilde{\mu} \circ R_e$,

Proof. From the hypothesis it follows that:

$0 = \inf(\nu, \mu - \nu)\ u = \inf\{\nu(a) + (\mu - \nu)(b);\ a + b = u, a, b \geqslant 0\}$ which implies the existence of a sequence $0 \leqslant a_n \leqslant u$ such that $(\mu - \nu)\, a_n \leqslant 2^{-n}$ and $\nu(u - a_n) \leqslant 2^{-n}$, $n \geqslant 1$. Put:

$$e = \inf\{\sup(a_k;\ k \geqslant n);\ n \geqslant 1\}$$

Because ν and $\mu - \nu$ are order σ-continuous and positive it follows that:

$$(1) \qquad 0 < (\mu - \nu)e < (\mu - \nu)\,(\sup(a_k;\ k \geqslant n))$$

and

$$(2) \qquad 0 \leqslant \nu(u - e) = \lim_n \nu\,(u - \sup(a_k; k \geqslant n)) \leqslant \lim_n \nu(u - a_n) = 0.$$

Let $a \in \mathcal{A}$, $0 \leqslant a \leqslant \inf(e, u - e)$. Then $\nu(a) = (\mu - \nu)\ (a) = 0$ and thus:

$$\mu(\inf(e, u - e)) \leqslant (\mu - \nu)\,(\inf(e, u - e) + \nu(\inf(e, u - e)) = 0.$$

We pass now to the proof of (iii). First, let us denote by $\tilde{\nu}$ a positive linear extension of ν to $\tilde{\mathcal{A}}$ such that $0 \leqslant \tilde{\nu} \leqslant \tilde{\mu}$. See Proposition 1.3 above. Then $\tilde{\nu}$ is order σ-continuous and because of (2) we have that $\tilde{\nu}((u - e) * x) = 0$ for all $x \in \mathcal{A}$. In fact, if $|x| \leqslant \gamma u$ then:

$$|\tilde{\nu}((u - e) * x)| \leqslant \tilde{\nu}((u - e) * |x|) \leqslant \gamma\nu(u - e) = 0.$$

By Theorem 1.1 above it follows that each positive $x \in \mathcal{A}$ is the l.u.b. of an increasing sequence of elements of the form considered above and thus our assertion is a consequence of the fact that $\tilde{\nu}$ is order σ-continuous.

In a similar way we can prove that

$$\tilde{\nu}(x * (u - e)) = (\tilde{\mu} - \tilde{\nu})\,(e * x) = (\tilde{\mu} - \tilde{\nu})\ (x * e) = 0$$

for every $x \in \mathcal{A}$ and thus:

$$\nu(x) = \tilde{\nu}((u - e) * x) + \tilde{\nu}(e * x) = \tilde{\nu}(e * x) = \tilde{\mu}(e * x) = \tilde{\mu}(x * e) = 0$$

for every $x \in \mathcal{A}$, q.e.d.

2.3 LEMMA. *If $\mathcal{A}$ is order σ-complete and the bilinear mapping $(a, b) \to \tilde{\mu}(a * b)$ is order σ-continuous, then for each $\nu \in \mathcal{A}^*$, with $|\nu| \leqslant \gamma\mu$ there exists an $a \in \mathcal{A}$ such that $|a| \leqslant \gamma u$ and*

$$\nu(x) = \tilde{\mu}(a * x) = \tilde{\mu}(x * a)$$

for every $x \in \mathcal{A}$.

Proof. Clearly it suffices to consider only the case when $0 \leqslant \nu \leqslant \mu$. Then by Theorem 1.1 above ν is the l.u.b. of an increasing sequence of positive simple elements $S_n \in [\mu]$. By using Lemma 2.2 above we check the existence of an increasing sequence of positive elements $s_n \in \mathscr{A}$ such that $0 \leqslant s_n \leqslant u$ and

$$S_n(x) = \tilde{\mu}(s_n * x) = \tilde{\mu}(x * s_n)$$

for every $x \in \mathscr{A}$. Put :

$$a = \sup \{s_n ;\ n \geqslant 1\}.$$

Then $0 \leqslant a \leqslant u$ and for each positive $x \in \mathscr{A}$ we have :

$$\nu(x) = (\sup S_n)(x) = \sup S_n(x) = \lim \tilde{\mu}(s_n * x) = \lim \tilde{\mu}(x * s_n) =$$
$$= \tilde{\mu}(a * x) = \tilde{\mu}(x * a), \text{ q.e.d.}$$

In order to state our main result we need a definition. If Z is an order σ-complete Banach lattice, a closed sublattice $Y \subset Z$ is said to be σ-minimal if Y is order σ-complete and $y_n \downarrow 0$ in Y implies $y_n \downarrow 0$ in Z.

2.4 THEOREM. *Let $Q = (\mathscr{A}, *, \tilde{\mathscr{A}})$ be a quasi-algebra with unit $u > 0$ and let $\tilde{\mu} \in \tilde{\mathscr{A}}^*$, $\tilde{\mu} > 0$. If $\mathscr{H}$ denote an order σ-complete σ-minimal closed sublattice of $[u]$ which contains $\mathscr{A}$, then for each $\nu \in \mathscr{A}^*$, with $|\nu| \leqslant \gamma\mu$, there exists an $a \in \mathscr{H}$ suh that $|a| \leqslant \gamma u$ and :*

$$\nu(x) = \tilde{\mu}(a *_\lambda x) = \tilde{\mu}(x *_\lambda a)$$

for every $x \in \mathscr{A}$. Here $[u]$ denotes the band generated by u in $\mathscr{A}^{**}$.

Proof. Our result follows immediately from Lemma 2.3 above by embedding Q into $(\mathscr{H}, *_\lambda, \tilde{\mathscr{A}}^{**})$ and observing that each $\nu \in \mathscr{A}^*$ is order σ-continuous when considered as belonging to $\mathscr{H}^*$ q.e.d.

Under the assumptions of the above theorem there is defined a relation of equivalency on $\mathscr{H}$ as follows :

$$x \overset{\tilde{\mu}}{\sim} y \text{ if and only if } \tilde{\mu}(|x - y|) = 0.$$

The completion of the quotient space $\mathscr{H}/\tilde{\mu}$ with respect to the following norm :

$$\|x\|_{\tilde{\mu}} = \sup_{\substack{\|y\| \leqslant 1 \\ y \in \mathscr{A}_+}} \tilde{\mu}(|x| *_\lambda y)$$

is a Banach lattice that will be denoted by $L(\tilde{\mu})$.

There is defined also a canonical mapping

$$V_{\mathscr{H}, \tilde{\mu}} : \mathscr{H}/\tilde{\mu} \to [\mu]$$

given by

$$V_{\mathscr{H}, \tilde{\mu}}(\hat{x}) = \tilde{\mu} \circ L_x^{**} \quad x \in \hat{x}$$

and it is clear that this mapping can be extended by continuity to $L(\tilde{\mu})$.

We can restate Theorem 2.4 above in terms of $V_{\mathscr{H},\tilde{\mu}}$ as follows:

2.5 THEOREM.

(i) *The image of* $V_{\mathscr{H},\tilde{\mu}}$ *contains all functionals* $\nu \in [\mu]$ *with* $|\nu| \leqslant \gamma\mu$.

(ii) *If* $\mathscr{A}^*$ *is weakly sequentially complete then* $V_{\mathscr{H},\tilde{\mu}}$ *extends to a lattice isometry from* $L(\tilde{\mu})$ *onto* $[\mu]$ *if, and only if,*

$$\tilde{\mu}(|h| *_\lambda a) = \sup_{\substack{|b| \leqslant a \\ b \in \mathscr{A}}} |\tilde{\mu}(h *_\lambda b)|$$

for every $h \in \mathscr{H}$, $a \in \mathscr{A}$.

Proof. Clearly only (ii) needs to be motivated. Because $\mathscr{A}^*$ is weakly sequentially complete then the topology of $\mathscr{A}^*$ is order σ-continuous i.e.

$$x_n^* \downarrow 0 \text{ (in order) implies } \|x_n^*\| \to 0$$

See [4] or [6] for details. By combining this remark with Theorem 1.1 above we obtain that the linear subspace generated by all quasi-units of $[\mu]$ is dense in $[\mu]$. Now if we assume that:

$$V_{\mathscr{H},\tilde{\mu}}(|h|) = |V_{\mathscr{H},\tilde{\mu}}(h)|$$

for every $h \in \mathscr{H}$, then $V_{\mathscr{H},\tilde{\mu}}$ is an isometry that maps quasi-units into quasi-units (combine Lemma 2.2 with the fact that $V_{\mathscr{H},\tilde{\mu}}$ is one-to-one) and our result follows.

2.6 REMARK. If the canonical mapping $V_{\mathscr{H},\tilde{\mu}}$ is one-to-one then:

$$\tilde{\mu}(x *_\lambda y) = \tilde{\mu}(y *_\lambda x)$$

for all $x, y \in \mathscr{H}$.

In fact, let $h \in \mathscr{H}$ with $|h| \leqslant \gamma u$. By Lemma 2.3 above h is the only element of $\mathscr{H}$ such that

$$\tilde{\mu}(h *_\lambda x) = \tilde{\mu}(y *_\lambda x) = \tilde{\mu}(x *_\lambda y)$$

for all $x \in \mathscr{A}$. Particularly this is the case when h is a simple element. The case of an arbitrary $h \in \mathscr{H}$ follows from Theorem 1.1 above and the order continuity of $\tilde{\mu}$ regarded as an element of $\tilde{\mathscr{A}}^{***}$.

3. EXAMPLES

Let S be a compact Hausdorff space and let $\mathscr{H}$ be the Banach lattice of all Borel measurable bounded functions $f: S \to R$. If μ denotes a positive Radon measure on S then $L(\mathscr{H}, \mathscr{H}') = L^1(\mu)$ and $|f|\mu = |f\mu|$ for every $f \in \mathscr{H}$. By Theorem 2.5(ii) above it follows that the canonical mapping $V_{\mathscr{H},\tilde{\mu}}$ extends to a lattice isometry from $L^1(\mu)$ onto $[\mu]$ and this fact is nothing but the classical Lebesgue-Radon-Nikodym theorem.

If μ denotes a positive Haar measure on a locally compact group G then $L^1(\mu)$ can be endowed with a structure of a Banach algebra in which the multiplication is the product of convolution :

$$(x * y)\, t = \int x(t - s)\, y(s)\, d\mu(s).$$

This algebra has a multiplicative unit if and only if G is discrete and generally this unit is not a Freudenthal unit.

3.1 PROPOSITION. *Let μ be a positive Radon measure. Then the Banach lattice $L^1(\mu)$ has a structure of quasi-algebra with unit only if* dim $L^1(\mu) < \infty$.

Proof. Clearly, we can assume that $L^1(\mu)$ is also separable. Let λ be the functional on $L^1(\mu)$ associated to $1 \in L^\infty(\mu)$. If $(L^1(\mu), *, \tilde{\mathscr{A}})$ is a quasi-algebra with unit and $\mathscr{H} = L^1(\mu)$ then the canonical mapping $V_{\mathscr{H},\lambda} : L^1(\mu) \to L^\infty(\mu)$ is onto (see Theorem 2.5 above) and thus $L^\infty(\mu)$ must be separable, which implies (see [2] Theorem 9, Cor. 2) that dim $L^\infty(\mu) < \infty$, q.e.d.

If μ is a positive Radon measure on a compact Hausdorff space S then $(L^2(\mu), *, L^1(\mu))$ constitutes a quasi-algebra with unit $1 \in L^2(\mu)$, where $*$ denotes the point-wise multiplication. Put $\lambda(f) = \int f\, d\mu$ for all $f \in L^2(\mu)$ and $\mathscr{H} = L^2(\mu)$. Then $L(\lambda) = L^2(\mu)$ and the canonical mapping $V_{\mathscr{H},\lambda}$ establishes an order isometry from $L^2(\mu)$ onto $(L^2(\mu))^*$; this coincides with the usual characterization (due to F. Riesz) for the conjugate of a Hilbert space.

4. THE VECTOR CASE

In the sequel $(\mathscr{A}, *, \tilde{\mathscr{A}})$ will denote a quasi-algebra with unit and X will denote a Banach space. In addition, $\mathscr{A}$ is assumed to have the approximation property in the sense of Grothendieck [3].

Let $\tilde{\mu} \in \tilde{\mathscr{A}}^*, \tilde{\mu} > 0$ and let $\mu = \tilde{\mu} \mid \mathscr{A}$. We shall prove the following Lebesgue-Radon-Nikodym type theorem that gives information on the vector space $N_\mu(\mathscr{A}, X)$ of all nuclear operators $T : \mathscr{A} \to X$, $T \ll \mu$.

4.1 THEOREM. $N_\mu(\mathscr{A}, X) = \mathscr{A}^* \hat{\otimes} X$.

Here the cap means the completion in the projective topology i.e., *the finest locally convex topology on* $[\mu] \otimes X$ *which makes continuous the canonical mapping* :

$$[\mu] \times X \to [\mu] \otimes X.$$

Proof. Because $\mathscr{A}$ has the approximation property we have $N(\mathscr{A}, X) = \mathscr{A}^* \hat{\otimes} X$ and thus it remains to prove the following inclusion :

$$N_\mu(\mathscr{A}, X) \subset [\mu] \hat{\otimes} X.$$

Let $T \in N_\mu(\mathscr{A}, X)$. We can extend T to $C(S)$ as a nuclear operator, S denoting the unit ball of $\mathscr{A}^*$. Use Hahn-Banach extension theorem and Theorem 1 in [3]. Then there exists an $x^* \in X^*$ such that $T \ll |x^* \circ T|$, which follows easily by combining the following two results :

(i) Let $m : \mathscr{T} \to X$ be a σ-additive measure defined on a Boolean σ-algebra. There is defined an $x^* \in X^*$ such that $m \ll |x^* \circ m|$. See [7] for details.

(ii) Let $U \in \mathscr{L}(C(S), X)$ be a compact operator and let μ be a positive Radon measure on S such that the measure m, canonically associated to U is absolutely continuous with respect to μ. Then $U \ll \mu$ in the sense of the definition in Introduction. See [6] for details.

Considered as a nuclear operator on $C(S)$, T can be represented as follows :

$$T = \sum \mu_n \otimes x_n,$$

where $x_n \in X$, $\sum \|x_n\| < \infty$ and $\{\mu_n\}_n$ is an equi-continuous sequence of scalar Radon measures on S. Moreover $\mu_n = \mu_{n,1} + \mu_{n,2}$, where $\{\mu_{n,1}\}_n$ is an equi-continuous sequence of Radon measures on S such that $\mu_{n,1} \ll |x^* \circ T|$ and $\mu_{n,2}$ are all singular with respect to $x^* \circ T$. We have :

$$T - \sum \mu_{n,1} \otimes x_n = \sum \mu_{n,2} \otimes x_n.$$

The left side is equivalent to a measure on S which is absolutely continuous with respect to $x^* \circ T$. On the contrary, the right side is a singular measure with respect to $x^* \circ T$ and thus :

$$T = \sum \mu_{n,1} \otimes x_n.$$

Because $T \ll \mu$, it follows that $x^* \circ T \ll \mu$, which in turn implies that $\mu_{n,1} \ll \mu$ for all $n \geqslant 1$, q.e.d.

4.2 Under the assumptions of Theorem 2.5 (ii) we obtain an algebraic isomorphism :

$$V_{\mathscr{H},\tilde{\mu}} \hat{\otimes} 1_X : L(\tilde{\mu}) \hat{\otimes} X \tilde{\to} N_\mu(\mathscr{A}, X)$$

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